COMPUTING MIXED VOLUME AND ALL MIXED CELLS IN QUERMASSINTEGRAL TIME.

GREGORIO MALAJOVICH

1. Mixed volume and Quermassintegrals

The *mixed volume* of the n-tuple of convex bodies $(\mathcal{A}_1, \dots, \mathcal{A}_n)$, $\mathcal{A}_i \subset \mathbb{R}^n$ is defined as

$$V(\mathcal{A}_1, \dots, \mathcal{A}_n) \stackrel{\text{def}}{=} \frac{1}{n!} \frac{\partial^n}{\partial t_1 \partial t_2 \cdots \partial t_n} \text{Vol}(t_1\mathcal{A}_1 + \cdots + t_n\mathcal{A}_n)$$

where $t_1, \dots, t_n \geq 0$ and the derivative is taken at $t = 0$. The normalization factor $1/n!$ ensures that $V(\mathcal{A}, \dots, \mathcal{A}) = \text{Vol}(\mathcal{A})$. In that sense the mixed volume generalizes the usual volume.

The definition above is still valid if we allow the $\mathcal{A}_i$ to have empty volume, so in this paper we will only require the $\mathcal{A}_i$ to be closed convex sets.

The mixed volume was introduced by Minkowski (1901) for $n = 3$. If $\mathcal{A} \subset \mathbb{R}^3$ is a convex body, the Steiner formula

$$\text{Vol}(\mathcal{A} + \epsilon B^3) = \text{Vol}(\mathcal{A}) + S\epsilon + \pi K \epsilon^2 + \frac{4}{3}\pi\epsilon^3$$

implies that

$$V(\mathcal{A}, \mathcal{A}, B^3) = 3S \qquad \text{and} \qquad V(\mathcal{A}, B^3, B^3) = 3\pi K$$

where S is the total area of ∂A and K its total mean curvature (assuming ∂A is smooth). The invariants $V(\mathcal{A}, \mathcal{A}, B^3)$ and $V(\mathcal{A}, B^3, B^3)$ are known as *quermassintegrals*. I will speak here on an algorithm for mixed volume computation with complexity bounded in terms of *mixed* quermassintegrals.

2. Mixed volume and polynomial systems

A system of n-variate polynomial systems is a system of equation of the form $f_1(x) = \cdots f_m(x) = 0$ where

$$f_i(\mathbf{x}) = \sum_{\mathbf{a} \in A_i} f_{ia} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \tag{1}$$

and each A_i is a finite subset of $(\mathbb{Z}_{\geq 0})^n$. The set A_i is known as the *support* of A_i. If the points of A_i are allowed to be in $\mathbb{Z}^n$, then f_i is said to be a *Laurent polynomial*.

In this talk, I will only consider (Laurent) polynomial systems with the same number of equations and variables, so $m = n$.

Theorem 1. (Bernstein, 1975; Bernstein et al., 1976) *The generic number of roots in $(\mathbb{C}^\times)^n$ of the system* (1) *is*

$$n! V(\text{Conv}(A_1), \dots, \text{Conv}(A_n)).$$

The case $n = 1$ is attibuted to Newton. If $A_1 = \cdots = A_n$, the system is said to be *unmixed*. The case of unmixed systems for $n = 2$ is due to Minding (2003) and the general unmixed case to Kušnirenko (1976).

3. Nonlinear homotopy and tropicalization

Huber and Sturmfels (1995) suggested solving such systems by a nonlinear homotopy, that amounts to

$$f_i(\mathbf{x}, t) = \sum_{\mathbf{a} \in A_i} f_{ia} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} t^{-b_i(a)} \quad , 1 \leq i \leq n.$$

where $b_i(a)$ is a random lifting. The initial system is obtained at the limit when $t \to \infty$, and amounts to finding something called the 'lower mixed facets' associated to the lifting b. To better understand what the nonlinear homotopy does, it is convenient to change coordinates:

$$t = e^\tau, \qquad x_i = e^{\tau \xi_i}.$$

Now

$$f_i(\xi, t) = \sum_{\mathbf{a} \in A_i} f_{ia} e^{\tau(\mathrm{a}\xi - b_i(\mathrm{a}))}$$

The *Legendre transform* of $\mathbf{a} \mapsto b_i(\mathbf{a})$ is a function λ_i on the variable ξ, given by

$$\lambda_i(\xi) = \max_{\mathbf{a} \in A_i} \mathbf{a}\xi - b_i(\mathbf{a}).$$

Under this new notation,

$$f_i(\xi, t) = e^{\tau \lambda_i(\xi)} \sum_{\mathbf{a} \in A_i} f_{ia} e^{\tau(\mathrm{a}\xi - b_i(\mathrm{a}) - \lambda_i(\xi))}$$

When $\tau \to \infty$, the equations can only be satisfied if the maximum of $\mathbf{a}\xi - b_i(\mathbf{a})$ is attained twice for each equation. This looks like a combinatorial problem, but in modern language it is a problem of *tropical algebraic geometry*, or algebraic geometry over the tropical semi-ring $(\mathbb{R} \cup \{-\infty\}, +, \max)$. See `?Mclagan-Sturmfels` () for a texbook on the subject.

4. The results

The algorithm AllMixedCells(Malajovich, 2015a) computes the mixed volume

$$V = V(\text{Conv}(A_1), \cdots, \text{Conv}(A_n))$$

together with the initial solutions, that is the zero-dimensional tropical variety of solution points to the limiting system when $\tau \to \infty$.

For a certain time, computing the starting system was a bottleneck for homotopy based polynomial solving software. Later breakthroughs by Gao and Li (2000), Li and Li (2001), Gao et al. (2005), Mizutani et al. (2007), Lee and Li (2011), and Chen et al. (2014) provided efficient practical implementations through enumerative algorithms. However, the complexity properties of those algorithms are not well understood.

The algorithm AllMixedCells is geometric in nature, and this will allow for a complexity bound in terms of mixed volumes. The input for the algorithms is the support $(A_1, \dots, A_n)$ of a system of polynomial equations such as (2).

Talk given at the ICIAM-2015, Beijing, Aug 13, 2015

The author was partially supported by CNPq and CAPES (Brazil). Substantial part of this work was done while visiting the Simons Institute for Theoretical Computer Science at the University of California at Berkeley, funded by a CAPES *estágio sênior* fellowship.

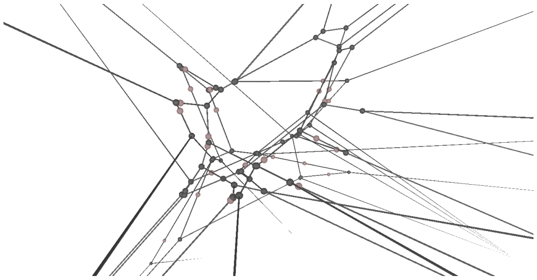

FIGURE 1. The Cohn 3 polynomial system.

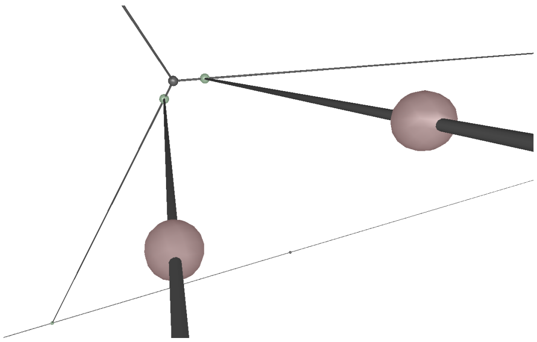

FIGURE 2. The Cyclic 3 polynomial system.

Let $\mathcal{A}_i = \text{Conv}(A_i)$, $\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_n$ and let B^n be the unit n-ball of radius 1. Let $d_i = \dim \text{Conv}(A_i)$. Let $V_i = d_i! \text{Vol}_{d_i}(\text{Conv}(A_i))$ be the generic root bound of an unmixed polynomial system of support A_i. Let $0 \le E_i < \#A_i$ be numbers defined in the paper.

Theorem 2. *With probability one, the algorithm* ALLMIXEDCELLSFULL *computes the mixed volume and produces all the initial points in time bounded by $O(T + T')$ arithmetic operations, where*

$$T = (\sum_{i=2}^{d} v_i) \left(n^2 \sum_{i=1}^{n} E_i + \log \sum_{i=2}^{n} v_i \right), \tag{2}$$

$$T' = (\max V_i) \left(n^2 \sum_{i=1}^{n} \#A_i + \log \max_{i=1,\ldots,n} V_i \right), \tag{3}$$

and

(a) With probability one,

$$v_d \le n! V(\mathcal{A}_1, \ldots, \mathcal{A}_{d-1}, \mathcal{A}, B^n, \ldots, B^n).$$

(b) On average,

$$\bar{v}_d \le \frac{n!}{2^{n-d}} V(\mathcal{A}_1, \ldots, \mathcal{A}_{d-1}, \mathcal{A}, B^n, \ldots, B^n).$$

5. THE MAIN IDEAS

(1) If one removes one of the tropical equations, one obtains a *tropical curve*, that is a set of segments and half-lines. Connected components of this graph can be explored by standard techniques.
(2) The tropical line can fail to be connected. To find one point of each connected component, one proceeds by induction: intersect the line with a generic hyperplane, and then drop one equation more, and so on.
(3) Because of heuristical considerations, those hyperplanes are picked at infinity.
(4) All together, one needs to explore a certain graph (union of the tropical curves and connections between them).
(5) The algorithm was implemented and parallelized. It is available as free software under GPL. (Malajovich, 2015b)
(6) A speed-up was obtained by using a random path strategy.

REFERENCES

Bernstein, D. N. 1975. *The number of roots of a system of equations*, Funkcional. Anal. i Priložen. **9**, no. 3, 1–4 (Russian). MR0435072 (55 #8034)

Bernstein, D. N., A. G. Kušnirenko, and A. G. Hovanskiĭ. 1976. *Newton polyhedra*, Uspehi Mat. Nauk **31**, no. 3(189), 201–202 (Russian). MR0492376 (58 #11500)

Chen, Tianran, Tsung-Lin Lee, and Tien-Yien Li. 2014. *Mixed volume computation in parallel*, Taiwanese J. Math. **18**, no. 1, 93–114, DOI 10.11650/tjm.18.2014.3276. MR3162115

Gao, Tangan and T. Y. Li. 2000. *Mixed volume computation via linear programming*, Taiwanese J. Math. **4**, no. 4, 599–619. MR1799755 (2001j:65026)

Gao, Tangan, T. Y. Li, and Mengnien Wu. 2005. *Algorithm 846: MixedVol: a software package for mixed-volume computation*, ACM Trans. Math. Software **31**, no. 4, 555–560, DOI 10.1145/1114268.1114274. MR2272345 (2007j:65059)

Huber, Birkett and Bernd Sturmfels. 1995. *A polyhedral method for solving sparse polynomial systems*, Math. Comp. **64**, no. 212, 1541–1555, DOI 10.2307/2153370. MR1297471 (95m:65100)

Kušnirenko, A. G. 1976. *Newton polyhedra and Bezout's theorem*, Funkcional. Anal. i Priložen. **10**, no. 3, 82–83. (Russian). MR0422272 (54 #10263)

Lee, Tsung-Lin and Tien-Yien Li. 2011. *Mixed volume computation in solving polynomial systems*, Randomization, relaxation, and complexity in polynomial equation solving, Contemp. Math., vol. 556, Amer. Math. Soc., Providence, RI, pp. 97–112, DOI 10.1090/conm/556/11009. MR2882664 (2012k:52019)

Li, T. Y. and Xing Li. 2001. *Finding mixed cells in the mixed volume computation*, Found. Comput. Math. **1**, no. 2, 161–181, DOI 10.1007/s102080010005. MR1830034 (2002b:65030)

Maclagan, Diane and Bernd Sturmfels. 2015. *Introduction to Tropical Geometry*, American Mathematical Society, Providence, RI.

Malajovich, Gregorio. 2015a. *Computing mixed volume and all mixed cells in quermassintegral time*, available at `http://arxiv.org/abs/1412.0480`.

______. 2015b. *Polynomial System Solver 5 pre-release 1*.

Minding, Ferdinand. 2003. *On the determination of the degree of an equation obtained by elimination*, Topics in algebraic geometry and geometric modeling, Contemp. Math., vol. 334, Amer. Math. Soc., Providence, RI, pp. 351–362. Translated from the German (Crelle, 1841)and with a commentary by D. Cox and J. M. Rojas. MR2039981 (2004m:01022)

Minkowski, Hermann. 1901. *Sur les surfaces convexes fermées*, C.R. Acad.Sci., Paris **132**, 21–24.

Mizutani, Tomohiko, Akiko Takeda, and Masakazu Kojima. 2007. *Dynamic enumeration of all mixed cells*, Discrete Comput. Geom. **37**, no. 3, 351–367, DOI 10.1007/s00454-006-1300-9. MR2301523 (2008b:52012)

DEPARTAMENTO DE MATEMÁTICA APLICADA, INSTITUTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO. CAIXA POSTAL 68530 21941-909 RIO DE JANEIRO - RJ - BRASIL

E-mail address: `gregorio.malajovich@gmail.com`